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Exact Lattice Supersymmetry at the Quantum Level for N=2 Wess-Zumino models in Lower Dimensions

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We have recently proposed a new lattice SUSY formulation which has exact lattice supersymmetry for Wess-Zumino models in one and two dimensions for all N=2 supercharges. This formulation is non-local in the coordinate space but the difference operator satisfies the Leibniz rule on the newly defined star product. Here we show that this lattice supersymmetry is kept exact at the quantum level by investigating Ward-Takahashi identities up to two loop level.

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1. Introduction

There are two major difficulties in constructing exact lattice SUSY formulation for all super charges:
1) The difference operator does not satisfy the Leibniz rule.
2) For massless lattice fermions species doublers of chiral fermions usually appear.
If we replace the differential operator by a difference operator in the SUSY algebra, lattice SUSY is broken at the algebraic level since the SUSY generators satisfy Leibniz rule while the difference operator does not follow to the Leibniz rule. Secondly if we put massless fermions on the lattice species doublers of the chiral fermion appear: an unavoidable consequence of the NO-GO theorem of chiral fermions on the lattice. In supersymmetry the number of boson degrees of freedom and that of fermions should be the same, and thus this chiral fermion doublers break the balance of degrees of freedom between the bosons and fermions. Thus lattice supersymmetry will be broken with the naive version of lattice fermion formulation. Even if we use the recently proposed chiral fermion formulation satisfying Ginzberg-Wilson relation, the treatment of fermions and bosons cannot be exactly the same leading to a breaking of exact lattice supersymmetry. It has recently been pointed out that the item 1) is in fact a NO-GO for local lattice formulation of supersymmetry[1].

With the aim of solving these difficulties we proposed the formulation of ref. [2][3]. For the problem 1) we identify the momentum representation of a symmetric lattice difference operator as a lattice momentum and impose the conservation of the lattice momenta for products of fields in the momentum representation. The importance of the lattice momentum conservation was noticed by the very first paper of lattice SUSY[4]. In solving the problem 2) we identify the species doublers as super partner particles in the same super multiplet. To keep the balance for the equal treatment of fermions and bosons we introduce the species doubler counter part for bosons. We briefly explain the lattice SUSY formulation N=2 Wess-Zumino model in two dimensions, which has exact lattice SUSY[3]. We explicitly show that the exact SUSY is kept at the quantum level by explicitly examining the Ward-Takahashi (WT) identities up to two loop level. One dimensional formulation of Wess-Zumino model which has exact lattice SUSY is given in [2].

2. D=N=2 Wess-Zumino action

$N=2$ extended supersymmetry algebra in two dimensions is given by

$$\{Q_{\alpha i}, Q_{\beta j}\} \;=\; 2\delta_{ij}(\gamma^{\mu})_{\alpha\beta}\, i\partial_{\mu}, \tag{2.1}$$

where we may use an explicit representation of Pauli matrices for $\gamma^{\mu} = \{\sigma^3, \sigma^1\}$. By going to the light cone directions this two dimensional $N=2$ algebra can be decomposed into the direct sum of two one dimensional $N=2$ algebra :

$$\{Q_{\pm}^{(i)}, Q_{\pm}^{(j)}\} \;=\; 2\delta^{ij} i\partial_{\pm}, \qquad \{\text{others}\} = 0, \tag{2.2}$$

where

$$Q_{\pm}^{(j)} \;=\; \frac{Q_{1j} \pm iQ_{2j}}{\sqrt{2}}, \qquad \partial_{\pm} = \partial_1 \pm i\partial_2, \tag{2.3}$$

Here we have introduced the following light cone coordinates

$$x_\pm = x_1 \pm i x_2, \qquad \partial_\pm = \frac{\partial}{\partial x_\pm}. \tag{2.4}$$

We can equivalently express the above algebra in a chiral form:

$$\{Q^{(+)}_\pm, Q^{(-)}_\pm\} \; = \; i\partial_\pm, \quad \{\text{others}\} = 0, \tag{2.5}$$

where

$$Q^{(+)}_\pm = \frac{Q^{(1)}_\pm + iQ^{(2)}_\pm}{2}, \quad Q^{(-)}_\pm = \frac{Q^{(1)}_\pm - iQ^{(2)}_\pm}{2}. \tag{2.6}$$

The corresponding momentum counterpart of the algebra is given by:

$$\{Q^{(+)}_\pm, Q^{(-)}_\pm\} \; = \; 2\sin\frac{ap_\pm}{2} \; \equiv \; \hat{p}_\pm. \tag{2.7}$$

In two dimensional N=2 SUSY algebra, we introduce four chiral fields $\Phi_A \equiv \{\Phi, \Psi_1, \Psi_2, F\}$ and the corresponding anti-chiral fields $\overline{\Phi}_A$. Each field Φ_A and $\overline{\Phi}_A$ has 4 species doublers. We can impose chiral and anti-chiral conditions which lead to the identification of the original fields with the species doubler fields [3]:

$$\begin{aligned}
\Phi_A(p_+,p_-) \; &= \; \Phi_A(\frac{2\pi}{a}-p_+,p_-) = \Phi_A(p_+,\frac{2\pi}{a}-p_-) = \Phi_A(\frac{2\pi}{a}-p_+,\frac{2\pi}{a}-p_-), \\
\overline{\Phi}_A(p_+,p_-) \; &= \; -\overline{\Phi}_A(\frac{2\pi}{a}-p_+,p_-) = -\overline{\Phi}_A(p_+,\frac{2\pi}{a}-p_-) = \overline{\Phi}_A(\frac{2\pi}{a}-p_+,\frac{2\pi}{a}-p_-)
\end{aligned} \tag{2.8}$$

with $\Phi_A \equiv \{\Phi, \Psi_1, \Psi_2, F\}$.

The kinetic term of the supersymmetric Wess-Zumino action can be written in a $Q-$exact form of action as in the continuum:

$$\begin{aligned}
S_K \; &= \; \int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} dp_+dp_-dq_+dq_-\delta(\hat{p}+\hat{q})Q^{(-)}_+Q^{(-)}_-Q^{(+)}_+Q^{(+)}_-\{\bar{\Phi}(p)\Phi(q)\} \\
&= \int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} dp_+dp_-dq_+dq_-\delta(\hat{p}+\hat{q})\Big[-4\bar{\Phi}(p)\sin\frac{aq_+}{2}\sin\frac{aq_-}{2}\Phi(q) - \bar{F}(p)F(q)\Big]. \\
&\quad +2\bar{\Psi}_2(p)\sin\frac{aq_+}{2}\Psi_2(q) + 2\bar{\Psi}_1(p)\sin\frac{aq_-}{2}\Psi_1(q)\Big].
\end{aligned} \tag{2.9}$$

The invariance of the action S_K under all the supersymmetry transformations generated by $Q^{(\pm)}_\pm$ is assured by the algebra of (2.7) whose component representation is given in Tables 1 and 2 and by the momentum conservation for the lattice momentum: $\hat{p} = 2\sin\frac{ap}{2}$,

$$\delta(\hat{p}+\hat{q}) \equiv \prod_{i=\pm}\frac{1}{2}\left[\delta(p_i+q_i) + \delta(p_i - q_i + \frac{2\pi}{a})\right], \qquad (\mathrm{mod}\frac{4\pi}{a}). \tag{2.10}$$

	$Q_+^{(+)}$	$Q_+^{(-)}$	$Q_-^{(+)}$	$Q_-^{(-)}$
$\Phi(p)$	$i\Psi_1(p)$	0	$i\Psi_2(p)$	0
$\Psi_1(p)$	0	$-2i\sin\frac{ap_+}{2}\Phi(p)$	$-F(p)$	0
$\Psi_2(p)$	$F(p)$	0	0	$-2i\sin\frac{ap_-}{2}\Phi(p)$
$F(p)$	0	$2\sin\frac{ap_+}{2}\Psi_2(p)$	0	$-2\sin\frac{ap_-}{2}\Psi_1(p)$

Table 1: Chiral $D=N=2$ supersymmetry transformation

	$Q_+^{(+)}$	$Q_+^{(-)}$	$Q_-^{(+)}$	$Q_-^{(-)}$
$\bar\Phi(p)$	0	$i\bar\Psi_1(p)$	0	$i\bar\Psi_2(p)$
$\bar\Psi_1(p)$	$-2i\sin\frac{ap_+}{2}\bar\Phi(p)$	0	0	$-\bar F(p)$
$\bar\Psi_2(p)$	0	$\bar F(p)$	$-2i\sin\frac{ap_-}{2}\bar\Phi(p)$	0
$\bar F(p)$	$2\sin\frac{ap_+}{2}\bar\Psi_2(p)$	0	$-2\sin\frac{ap_-}{2}\bar\Psi_1(p)$	0

Table 2: anti-chiral $D=N=2$ supersymmetry transformation

Interaction terms can be obtained by Q-exact form of the following action:

$$\begin{aligned} S_n &= \int \prod_{j=1}^n d^2p_j V_n(p) Q_+^{(+)} Q_-^{(+)} \{\Phi(p_1)\Phi(p_2)\cdots\Phi(p_n)\} + \text{h.c.} \\ &= \int \prod_{j=1}^n d^2p_j V_n(p) n \left[iF(p_1)\prod_{j=2}^n \Phi(p_j) + (n-1)\Psi_2(p_1)\Psi_1(p_2)\prod_{j=3}^n \Phi(p_j)\right] + \text{h.c.}, \end{aligned} \tag{2.11}$$

where $V_n(p)$ is

$$V_n(p) = a^{2n}\, g_n\, G_n(p)\, \delta^{(2)}\left(\sin\frac{ap_1}{2} + \sin\frac{ap_2}{2} + \cdots + \sin\frac{ap_n}{2}\right), \tag{2.12}$$

with $G_n(p)$ as appropriate momentum function which does not affect to the lattice SUSY invariance.

We assume that all fields satisfy the (anti-) chiral conditions (2.8), so that in each variable the contribution of the integration in the intervals $(-\frac{\pi}{a},\frac{\pi}{a})$ and $(\frac{\pi}{a},\frac{3\pi}{a})$ coincide and we get

$$\begin{aligned} S_K = 4\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} & dp_+dp_-dq_+dq_- \delta(p_++q_+)\delta(p_-+q_-)\left[-4\bar\Phi(p)\sin\frac{aq_+}{2}\sin\frac{aq_-}{2}\Phi(q)\right. \\ & \left. -\bar F(p)F(q) + 2\bar\Psi_2(p)\sin\frac{aq_+}{2}\Psi_2(q) + 2\bar\Psi_1(p)\sin\frac{aq_-}{2}\Psi_1(q)\right]. \end{aligned} \tag{2.13}$$

The mass term in momentum representation is given as:

$$S_2 = ma^2 \int \prod_{j=1}^2 d^2p_j\ \delta(\hat p_1 + \hat p_2)\,[iF(p_1)\Phi(p_2) + \Psi_2(p_1)\Psi_1(p_2)], \tag{2.14}$$

where the chiral conditions (2.8) are imposed.

The dimensionless chiral fields can be rescaled with powers of the lattice constant a to match the canonical dimensions of the component fields:

$$\Phi(p) \to a^{-2}\varphi(p), \qquad \Psi_i(p) \to a^{-\frac{3}{2}}\psi_i(p), \qquad F(p) \to a^{-1}f(p). \tag{2.15}$$

The anti-chiral fields are similarly rescaled. It is also necessary to rescale supercharges to recover correct canonical dimension: $Q_i^{(j)} \to a^{\frac{1}{2}}Q_i^{(j)}$. The kinetic term in momentum representation then reads:

$$S_k = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d\hat{p}^2}{(2\pi)^2}\left[-\overline{\varphi}(-p)\hat{p}_+\hat{p}_-\varphi(p) - \overline{f}(-p)f(p) + \overline{\psi}_1(-p)\hat{p}_-\psi_1(p) + \overline{\psi}_2(-p)\hat{p}_+\psi_2(p)\right], \tag{2.16}$$

where the dimensional lattice momentum is $\hat{p}_\pm = \frac{2}{a}\sin\frac{ap_\pm}{2}$.

3. Ward-Takahashi identities

The equivalence under the fields redefinition leads to the following identities:

$$\begin{aligned}\langle\mathscr{O}\rangle = \frac{1}{\mathscr{Z}}\int\mathscr{D}[\Phi]\mathscr{O}[\Phi]e^{i\mathscr{S}[\Phi]} &= \frac{1}{\mathscr{Z}}\int\mathscr{D}[\Phi']\mathscr{O}[\Phi']e^{i\mathscr{S}[\Phi']},\\ &= \frac{1}{\mathscr{Z}}\int\mathscr{D}[\Phi]\left(\mathscr{O}[\Phi]+\delta\mathscr{O}[\Phi]\right)e^{i\mathscr{S}[\Phi]+i\delta\mathscr{S}},\\ &= \langle\mathscr{O}\rangle + \langle\delta\mathscr{O}[\Phi]\rangle + \langle\mathscr{O}[\Phi]\delta\mathscr{S}[\Phi]\rangle + \cdots.\end{aligned} \tag{3.1}$$

where we assume that the functional measure is not anomalous under the symmetry.

If the action is invariant under the transformation: $\delta\mathscr{S}[\Phi] = 0$, we obtain the following identity:

$$\langle\delta\mathscr{O}[\Phi]\rangle = 0. \tag{3.2}$$

To find nontrivial relations between two point functions, we examine possible combinations of operators for $\mathscr{O}$. For example if we choose $\mathscr{O} = \phi\overline{\psi}_1$ and δ as lattice SUSY transformation of $Q_+^{(+)}$, we obtain

$$\langle\psi_1(p)\overline{\psi}_1(-p)\rangle + \hat{p}_+\langle\varphi(p)\overline{\varphi}(-p)\rangle = 0. \tag{3.3}$$

Tree propagators are given by

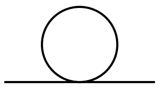
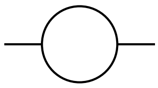
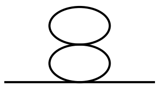
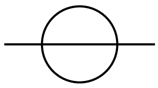
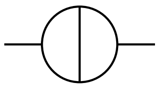
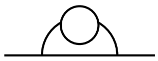
$$\langle\varphi(p)\overline{\varphi}(-p)\rangle_{\text{tree}} = \frac{-1}{D(\hat{p})}, \quad \langle\psi_1(p)\overline{\psi}_1(-p)\rangle_{\text{tree}} = \frac{\hat{p}_+}{D(\hat{p})}, \tag{3.4}$$

where $D(\hat{p}) = \hat{p}_+\hat{p}_- - m^2$. Apparently tree propagators (3.4) satisfy the identity (3.3), and it is consistent with the fact that the action is exactly invariant under the lattice supersymmetry at the classical level. We can choose other combinations of fields and lattice super charges for examining the W-T identities.

The basic structure of the loop contribution of corresponding diagrams to the two point function has the following form:

$$\langle\varphi(p)\overline{\varphi}(-p)\rangle_A = \langle\varphi(p)\overline{\varphi}(-p)\rangle_{\text{tree}} X_A(\hat{p}),$$
$$\langle\psi_1(p)\overline{\psi}_1(-p)\rangle_A = \langle\psi_1(p)\overline{\psi}_1(-p)\rangle_{\text{tree}} X_A(\hat{p}),$$

where $X_A(\hat{p})$'s are given as follows:

Loop diagram	$X_A(\hat{p})$
	0
	$-2g_3^2\dfrac{\hat{p}_+\hat{p}_- + m^2}{D(\hat{p})}I_1$
	0
	$-6g_4^2\dfrac{\hat{p}_+\hat{p}_- + m^2}{D(\hat{p})}I_2$
	$16m^2g_3^4\dfrac{2\hat{p}_+\hat{p}_- + m^2}{D(\hat{p})}I_3$
	$8g_3^4\dfrac{\hat{p}_+\hat{p}_- + m^2}{D(\hat{p})}I_4$

where

$$I_1 = \int \frac{d^2\hat{k}}{(2\pi)^2}\frac{1}{D(\hat{k})D(\hat{p}-\hat{k})}, \tag{3.5}$$

$$I_2 = \int \frac{d\hat{k}_1^2}{(2\pi)^2}\frac{d\hat{k}_2^2}{(2\pi)^2}\frac{1}{D(\hat{k}_1)D(\hat{k}_2)D(\hat{p}-\hat{k}_1-\hat{k}_2)}, \tag{3.6}$$

$$I_3 = \int \frac{d\hat{k}_1^2}{(2\pi)^2}\frac{d\hat{k}_2^2}{(2\pi)^2}\frac{1}{D(\hat{k}_1)D(\hat{k}_2)D(\hat{k}_1+\hat{p})D(\hat{k}_2+\hat{p})D(\hat{k}_1-\hat{k}_2)}, \tag{3.7}$$

$$I_4 = \int \frac{d^2\hat{k}_1 d^2\hat{k}_2}{(2\pi)^2(2\pi)^2}\frac{\hat{k}_1^2+m^2}{D(\hat{k}_1)^2D(\hat{k}_2)}\int\frac{d^2\hat{k}}{(2\pi)^2}\frac{1}{D(\hat{k})D(\hat{k}_1-\hat{k})} \tag{3.8}$$

Therefore the W-T identity of this particular combination is exactly satisfied up to the 2-loop level. We can show that the other combinations of the two point functions and SUSY transformations have the same structure as this example. In this way we may conclude that the W-T identities are satisfied exactly at the quantum level for all super charges. The details of the W-T identities calculations for D=N=2 Wess-Zumino model will be found in [5].

4. Discussions

In confirming the exact lattice SUSY invariance lattice momentum conservation plays a crucial role. This lattice momentum consevation defines a new type of $\star$-product of fields F and G:

$$(F \star G)(p) = \int d^2p_1 d^2p_2 F(p_1)G(p_2)\delta^{(2)}(\hat{p} - \hat{p}_1 - \hat{p}_2), \tag{4.1}$$

where the lattice momentum conservation is introduced. If we introduce standard momentum p conservation instead of the lattice momentum $\hat{p}$, the coordinate representation of the product of the function F and G leads to the standard product. However the coordinate representation of the $\star$-product with the lattice momentum leads to a non-local product of two functions. The details of $\star$-product can be found in [2] and [3]. It would be interesting to find a connection with this nonlocal nature of the $\star$-product and the noncommutative nature of link approach of lattice SUSY formulation[6] with Hopf algebraic lattice SUSY invariance[7].

One of the other characteristics of this $\star$-product is that the product is not associative. A given product, however, is well defined and thus the invariance of the lattice SUSY transformation is assured since SUSY transformation is linear with respect to fields. However non-associativity may be a problem when we try to extend this formulation to gauge theories since gauge transformations are nonlinear in fields. We will come back to this problem in future publication. Translational invariance is mildly broken since we use lattice momentum which is not periodic in itself. We can, however, show that it is recovered in the continuum limit[3].

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